B.sc(H) part 2 paper 3

Topic:Necessary and sufficient condition for a subgroup

Subject:Mathematics

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Theorem 1

Let G be a group and H a non-empty subset of G. Then H is a subgroup of G if, and only if (i) for all $a \in H$, $b \in H \Rightarrow ab \in H$, i.e. H is closed under the given operation and (ii) for all $a \in H$, the inverse of a i.e. $a^{-1} \in H$.

Proof : The proof of the theorem consists of two parts. The first part consists in showing that if H is a subgroup of G, the conditions (i) and (ii) are true, and the second part consists in showing that if the two conditions (i) and (ii) hold good then H is a group and hence a subgroup of G.

The conditions are necessary

The first part of the theorem follows from the fact that H is given to be a group under the group operation of G. Now since H is a group, therefore for all $a, b \in H$, we have $ab \in H$ and the first condition is satisfied. Also, for all $a \in H$, we have $a^{-1} \in H$ and the second condition is satisfied. Thus if H is a subgroup of G, the conditions (i) and (ii) are satisfied and the first part of the theorem is proved.

The conditions are sufficient

Now, to prove the second part, we must show that H is a group under the operation of G. We shall show that with the given two conditions, H satisfies all the four postulates of a group.

(i) We are given that for all $a, b \in H$, $ab \in H$ and hence the first postulate is satisfied.

(ii) H is associative because H is a subset of G which is associative.

(iii) If $x \in H$, then because of condition (ii), $x^{-1} \in H$ and the fourth postulate is satisfied.

(iv) Again, we have, by (i), $xx^{-1} \in H$, i.e. $e \in H$. Hence the identity element of H is necessarily e.

Thus, we see that H satisfies all the four postulates of a group and hence it is a subgroup.

The conditions (i) and (ii) of this theorem can be replaced by the following single condition.

THEOREM II.

G be a group and H a non-empty subset of G. Then H is a subgroup of G if and only if

$$[a \in H, b \in H] \Rightarrow ab^{-1} \in H.$$

Proof : The condition is sufficient

We shall first prove that the condition is sufficient. That is, we shall prove that if $a \in H, b \in H \Rightarrow ab^{-1} \in H$, then H is a group. In this connection we should remember that if $a \in H$, then $a \in G$ also, since H is a subset of G.

Given that $[a \in H, b \in H] \Rightarrow ab^{-1} \in H$.

Existence of Identity :

In this relation taking $b = a$, we get

$[a \in H, b \in H] \Rightarrow aa^{-1} \in H$, where a^{-1} is the inverse of a in G.

$\Rightarrow e \in H$, where e is the identity of G.

Hence e is an identity of H also.

Thus postulate 3 is satisfied.

Existence of Inverse :

Again, let $a \in H$. Then $[e \in H, a \in H] \Rightarrow e \cdot a^{-1} \in H$.

That is, $a^{-1} \in H$.

Thus if $a \in H$, then its inverse $a^{-1} \in H$.

Thus postulate 4 is satisfied.

Closure Property :

Now $[a \in H, b^{-1} \in H] \Rightarrow a(b^{-1})^{-1} \in H$;

$\Rightarrow ab \in H$.

Hence $[a \in H, b \in H] \Rightarrow ab \in H$

Thus postulate 1 is satisfied.

Associativity : The binary operation in G is associative and since it is a subset of G, it must be associative in H also. Thus postulate 2 is satisfied.

Hence the set H forms a group.

The condition is necessary

We shall prove that the condition is necessary. That is, we shall prove that if H is a group and $a, b \in H$, then

$$ab^{-1} \in H.$$

Given that H is a group and $b \in H$; then its inverse $b^{-1} \in H$ also. Therefore according to the first postulate $ab^{-1} \in H$.

since H is a group. Thus the above condition has been proved to be necessary.

Note : In the case of additive group, this single condition can be written down as $a \in H, \; b \in H \Rightarrow a - b \in H$.

: THEOREM III

A non-empty subset H of a finite group G is a subgroup of G iff

$$a \in H, \; b \in H \in ab \in H.$$

It should be noted that in this theorem only one condition viz, closure property is needed. (i.e. H is closed under multiplication)

The condition is necessary

Suppose H is a subgroup of G. Then H must be closed with respect to multiplication i.e. the composition in G. Therefore $a \in H, \; b \in H \Rightarrow ab \in H$

Hence the condition is necessary.

The condition is sufficient

It is given that H is closed with respect to multiplication i.e. $a \in H, \; b \in H \Rightarrow ab \in H$.

Let a be any element of H. Then by the given condition $a^2 = aa \in H, \; a^3 = aa^2 \in H, a^4 = aa^3 \in H$ and so on. Thus proceeding in this way, we get $a^n \in H$ where n is any positive integer.

Thus the infinite collection of elements $a, a^2, a^3, \ldots a^n, \ldots$ all belong to H. But H is a finite subset of G.

Therefore there must be repetitions in this collection of elements (if they are distinct, then H will not be finite set). Hence there must exist an identity $e \in H$ such that $a^n = e$ which will be also the identity element of G.

Also, $\quad a^{n-1} \in H$ i.e. $a^n \circ a^{-1} \in H$

i.e. $\quad e \circ a^{-1} \in H$

i.e. $\quad a^{-1} \in H$.

Thus $\quad a \in H \Rightarrow a^{-1} \in H$.

In this way we find that the two conditions of Theorem I of 3.2 hold good. Hence H is a subgroup of G.